Algebra

This is the last time it will really make sense to have "Algebra" as a separate section. Once you start A level, everything will involve algebra. You'll have algebra in your fractions, algebra in your surds, your equations will be trigonometric, and your circles will be defined by equations. Even your long division will be algebraic.

This makes it extremely important that you are confident with the algebra you already know. The new content you cover will take for granted that you are able to apply this knowledge in challenging situations you won't have seen before. You'll need to be comfortable with the algebraic tools you have, and be able to spot where you can use them; maybe you can factorise here to simplify this fraction... maybe you can solve this trigonometric equation if you notice that it includes a difference between two squares...

Expanding Polynomials

Polynomials appear throughout the AS and A level course. A polynomial is just an expression consisting of variables (like x or y), numbers and positive integer (whole number) powers. The terms in this expression can be added or subtracted. For example, $3x^2 + 2x - 4$ is a polynomial but $3x^{-4}$ is not.

Expanding (Multiplying) Brackets

You will need to be able to multiply out single, double and triple brackets.

To expand single brackets, multiply everything inside the bracket by the term on the outside.

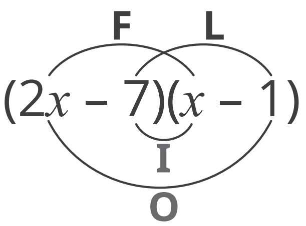
E.g. Expand $3(2x + 5y)$

$3(2x + 5y) = (3 \times 2x) + (3 \times 5y)$

$\qquad = \mathbf{6x + 15y}$

To expand double brackets, multiply everything in the first bracket by everything in the second. The **FOIL** (**f**irst, **o**uter, **i**nner, **l**ast) method can be helpful in ensuring you don't miss any terms.

E.g. Expand $(2x - 7)(x - 1)$



$(2x - 7)(x - 1) = (2x \times x) + (2x \times -1) + (-7 \times x) + (-7 \times -1)$

$\qquad = 2x^2 - 2x - 7x + 7$

$\qquad = \mathbf{2x^2 - 9x + 7}$

To expand triple brackets, begin by multiplying one pair of brackets, then multiply each term in the expanded expression by each term in the remaining bracket. You will need to be systematic in your approach so you don't lose any terms. A grid method can help.

E.g. Expand $(x - 1)(x + 2)(x + 3)$

$(x - 1)(x + 2)(x + 3) = (x^2 + 2x - x - 2)(x + 3)$

$\qquad = (x^2 + x - 2)(x + 3)$

	x^2	x	-2
x	x^3	x^2	$-2x$
3	$3x^2$	$3x$	-6

$(x^2 + x - 2)(x + 3) = x^3 + 3x^2 + x^2 + 3x - 2x - 6$

$(x - 1)(x + 2)(x + 3) = \mathbf{x^3 + 4x^2 + x - 6}$

Notice that the constant term in the expansion is the product (that means times) of the numerical part in the brackets. $1 \times -2 \times 3 = -6$. This is important for helping us to factorise cubic expressions later on, but can be a nice way to check whether your work is likely to be correct.

Your Turn:

Expand and simplify:

1. $5(2x - 7)$

2. $8x(2x + 3)$

3. $7a(3a + 2b - 4)$

4. $5(2x + 1) + 3(x + 4)$

5. $8y(y - 4) - 2y(3 - y)$

6. $(3x + 2)(x + 5)$

7. $(x - 4)(3x - 9)$

8. $(a + b)(b - c)$

9. $(3x + 2)^2$

10. $(x + 8)(2x + y - 4)$

11. $(x + 3)(x + 4)(x + 1)$

12. $(2x - 5)(x - 2)(x + 7)$

13. $(x + 1)^3$

14. $(x + 2)^2(x + 5)$

For answers, go to page 100.

Factorising

To *factorise* means to put an expression back into brackets. To do this, begin by taking out any common factors among the terms (that's where the word comes from!). Then divide each term by this number, or expression, to find the terms that go inside the brackets.

E.g. Factorise $8x^2 - 10x$

The common factor is $2x$, so $2x$ goes on the outside of the brackets. $8x^2 \div 2x = 4x$ and $10x \div 2x = 5$.

$8x^2 - 10x =$ $\mathbf{2x(4x - 5)}$

A common misconception is to think that, because an expression contains more than two terms or has a squared term in it, it must factorise into double brackets. This is not always true.

E.g. Factorise $20p^2q + 5pq^2 - 15pq$

The common factor is $5pq$, so $5pq$ goes on the outside of the brackets.
$20p^2q \div 5pq = 4p$, $5pq^2 \div 5pq = q$ and $-15pq \div 5pq = -3$.

$20p^2q + 5pq^2 - 15pq =$ $\mathbf{5pq(4p + q - 3)}$

You might even see an expression where the common factor is itself an expression.

E.g. Factorise $4(x + y) - p(x + y)$

The common factor is $(x + y)$. $4(x + y) \div (x + y) = 4$ and $-p(x + y) \div (x + y) = -p$.

$4(x + y) - p(x + y) =$ $\mathbf{(x + y)(4 - p)}$

Your Turn:

Factorise fully:

1. $12x + 15$

   ____

2. $27x - 18$

   ____

3. $10y^2 + 28y$

   ____

4. $14ab + 21a$

5. $32x + 40y - 24$

6. $10x^2y - 15xy^2$

7. $12a^3b^2 + 18a^2b^3 - 27ab^4$

8. $a(b + c) + 5(b + c)$

9. $x(y + 3) + 2(y + 3)$

10. $2r(a - 4) - p(a - 4)$

For answers, go to page 101.

Factorising Quadratic Expressions

Quadratic expressions are of the form $ax^2 + bx + c$, where $a \neq 0$. a and b are called the coefficients of x^2 and x respectively. A quadratic is a polynomial, but since these appear so often in the AS and A-level course, they get their own section!

Factorising: When $a = 1$

When $a = 1$, the expression is $x^2 + bx + c$. If it can be factorised, this sort of expression will go into two brackets, with an x at the front of each. To find the numerical part, find two numbers that multiply to give c and add to give b.

E.g. Factorise $x^2 + 2x - 15$

Find two numbers that multiply to give -15 and add to give 2. List the factors of 15 then deal with the signs. The factors of 15 are 1 and 15, or 3 and 5.

A negative multiplied by a positive is a negative so one number in each factor pair will have to be negative. To give a sum of positive 2, we choose -3 and 5.

$x^2 + 2x - 15 = \mathbf{(x - 3)(x + 5)}$

Check your work by expanding the brackets and checking you get the original expression.

Your Turn:

Factorise fully:

1. $x^2 + 7x + 10$
2. $x^2 + 12x + 20$
3. $x^2 + 4x - 21$
4. $x^2 - x - 6$
5. $x^2 - 13x + 30$
6. $x^2 - 10x + 25$

For answers, go to page 103.

Factorising: The Difference of Two Squares

If the expression consists of two square numbers separated by a minus sign then it can be factorised using the difference of two squares rule.

$a^2 - b^2 = (a + b)(a - b)$

E.g. Factorise $x^2 - 25$

$5^2 = 25$, so this becomes $\mathbf{(x + 5)(x - 5)}$

This works even if the coefficient of x is not 1 (as long as it's square) or if both terms are algebraic.

E.g. Factorise $16x^2 - 49y^2$

$(4x)^2$ is $16x^2$ and $(7y)^2$ is $49y^2$, so this becomes $\mathbf{(4x + 7y)(4x - 7y)}$

Your Turn:

Factorise fully:

1. $x^2 - 36$
2. $a^2 - 81$
3. $4x^2 - 9$
4. $25a^2 - b^2$
5. $9x^2 - 100y^2$
6. $x^4 - y^2$

For answers, go to page 103.

Factorising – When $a \neq 1$

This is a little more involved and, luckily, you will have a calculator that can do most of this for you. However, you still need to be able to factorise this sort of expression.

There are a number of techniques: one of which is observation; another involves reversing the process for expanding brackets.

For an expression of the form $ax^2 + bx + c$, begin by multiplying a and c to get ac. Then, find a pair of numbers whose product is ac and whose sum is b. Next, split up the middle term into two x terms with these numbers as their coefficients. Finally, factorise each *pair* of terms.

E.g. Factorise $2x^2 + 9x + 10$

$2 \times 10 = 20$

Two numbers that have a product of 20 and a sum of 9 are 4 and 5.

$2x^2 + 9x + 10 = 2x^2 + 4x + 5x + 10$

Factorise the first pair of terms and the second.

$2x^2 + 4x + 5x + 10 = 2x(x + 2) + 5(x + 2)$

Finally, fully factorise by taking out a factor of $(x + 2)$

$2x(x + 2) + 5(x + 2) =$ $\mathbf{(x + 2)(2x + 5)}$

Your Turn:

Factorise fully:

1. $2x^2 + 11x + 12$
2. $3x^2 + 26x + 35$
3. $4x^2 + 8x - 21$
4. $3x^2 - 19x + 20$

For answers, go to page 104.

Completing the Square

By completing the square, we can solve non-factorable quadratic equations, perform proofs and identify turning points on quadratic graphs. The completed square form for the expression $x^2 + bx + c$ is $(x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$. In other words, we halve the coefficient of x to find the numerical part inside the brackets. Then, square this and subtract it from the bracketed expression.

E.g. Write $x^2 - 10x + 3$ in the form $(x + m)^2 + n$, where m and n are integers.

Begin by halving the coefficient of x to find the value of m. Then, square this number and subtract it.

$-10 \div 2 = -5$

$x^2 - 10x + 3 = (x - 5)^2 - 5^2 + 3$

$= \mathbf{(x - 5)^2 - 22}$

When we have an expression in completed square form, we can find the turning point. A quadratic graph whose equation is $y = (x + m)^2 + n$ has a turning point at $(-m, n)$.

Take our previous example. The quadratic equation $y = x^2 - 10x + 3$ can be written as $y = (x - 5)^2 - 22$. This means that the quadratic graph has a turning point at (5, -22).

E.g. A curve is given by the equation $y = x^2 + 4x - 1$. Write the equation in the form $y = (x + m)^2 + n$. Hence, write down the coordinates of the turning point of this graph.

$4 \div 2 = 2$

$x^2 + 4x - 1 = (x + 2)^2 - 2^2 - 1$

$= \mathbf{(x + 2)^2 - 5}$

The turning point has coordinates (-2, -5).

Note that, if the coefficient of x^2 is not equal to 1, you will need to factorise part of the expression first.

E.g. $3x^2 + 6x + 5 = 3(x^2 + 2x) + 5$

$= 3((x + 1)^2 - 1^2) + 5$

$= 3(x + 1)^2 - 3 + 5$

$= \mathbf{3(x + 1)^2 + 2}$

Your Turn:

Write each equation in completed square form, and then find the coordinates of the turning point.

1. $y = x^2 + 8x + 23$

2. $y = x^2 - 6x + 1$

3. $y = x^2 + 4x - 6$

4. $y = x^2 + 3x + 9$

5. $y = x^2 - 5x - 8$

6. $y = 2x^2 + 12x + 7$

7. $y = 3x^2 + 12x + 2$

8. $y = 2x^2 + 6x + 23$

For answers, go to page 105.

Linear Equations and Inequalities

Solving Equations and Inequalities

To solve a linear equation or inequality, we apply a series of inverse operations to isolate the variable (usually the letter x). If there are any fractions or brackets in the equation, it is sensible to deal with those first, then find a way to collect all the letters on one side of the equation and all the numbers on the other.

Example 1

Solve $\frac{4x + 5}{3} = 2(x - 7)$

Begin by multiplying both sides by 3 to remove the denominator of the fraction.

$4x + 5 = 6(x - 7)$

Next, multiply out the brackets. Sometimes, you will be able to divide through by the number in front but here that doesn't help us!

$4x + 5 = 6x - 42$

Gather the letters on one side by subtracting $4x$.

$5 = 2x - 42$

Gather the numbers on the other side by adding 42.

$47 = 2x$

Divide through by 2. It can be sensible to leave the answer in a fraction in its simplest form.

$\boldsymbol{\frac{47}{2} = x}$

When working with inequalities, if we divide or multiply **both sides** by a negative number, we must remember to reverse the inequality symbol.

Example 2

Solve $-5 < 7 - 3x \leq 11$

Begin by subtracting 7 from all sides.

$-12 < -3x \leq 4$

Next, divide by -3 and reverse the inequality symbols.

$\mathbf{4 > x \geq -\frac{4}{3}}$

It is convention to write this answer as $\mathbf{-\frac{4}{3} \leq x < 4}$ with the lower number in the range appearing first.

Your Turn:

1. Solve the following equations:

   a. $8(2x + 3) = 24$

   b. $\frac{3x - 4}{2} = 5$

   c. $2(\frac{3(x + 1)}{5}) = 6$

   d. $4(2x - 5) = 3(x + 2)$

   e. $\frac{5x - 7}{x} = 9$

   f. $8 - \frac{3x}{2 + x} = 10$

2. Solve the following inequalities:

   a. $8x + 3 > 2(x + 5)$

   c. $7 \leq 4x + 5 < 19$

   b. $\frac{2x - 1}{7} \leq 3$

   d. $5(3 - 2x) \geq 1$

3. Find the set of solutions which satisfies the following inequalities:
   $8x \geq 5 - x$ and $-4 < 3x + 1 \leq 10$

For answers, go to page 106.