## Graphs

## Equations of Straight-Line Graphs

The general equation of a straight line is $y=m x+c$, where $m$ represents the gradient (steepness) of the line and $c$ is the value of the $y$-intercept. You might need to rearrange an equation to be able to calculate one of these.

## Example 1

Find the gradient and the coordinates of the $y$-intercept of the line with equation $3 x+2 y=6$.

Begin by rearranging to make $y$ the subject.
$2 y=6-3 x$
$y=3-\frac{3}{2} x$

The gradient is $-\frac{3}{2}$ and the $y$-intercept is 3 , so its coordinates are $(\mathbf{0}, \mathbf{3})$.

You can find the gradient of a straight line given any two points on that line. If those two points are $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the formula for the gradient of a straight line is $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$.

## Parallel and Perpendicular Lines

Two lines are parallel if their gradients are equal, and they are perpendicular if the product of their gradients is -1 . In other words, if $m_{1}$ and $m_{2}$ are the gradients of two lines, these lines are perpendicular if $m_{1} \times m_{2}=-1$.

## Example 2

Prove that the line passing through the points $(4,1)$ and $(2,6)$ is perpendicular to the line whose equation is $5 y=2 x+3$.

The gradient of the first line is $m=\frac{6-1}{2-4}=-\frac{5}{2}$
The equation of the second line can be written as $y=\frac{2}{5} x+\frac{3}{5}$, so its gradient is $\frac{2}{5}$

## $-\frac{5}{2} \times \frac{2}{5}=-1$ therefore the lines are perpendicular.

Note that it's sensible to leave any non-integer values in fractional form. Fractions are generally easier to work with and if the equivalent decimal is non-terminating then they also provide an exact solution.

If you know the gradient of a line and the coordinates of at least one point, you can find the equation. Substitute everything you know into the equation of the line and then solve for $c$.

## Example 3

Find the equation of the line which is parallel to the line with equation $y=4 x+2$ and which passes through (3, 7).

The gradient of the line given is 4 and the lines are parallel, so the gradient of our line is also 4 .

Substitute this value into the equation $y=m x+c$ to get $y=4 x+c$.

We know the line passes through $(3,7)$, so substitute $x=3$ and $y=7$ into this equation and solve for $c$.
$7=4 \times 3+c$
$7=12+c$
$c=7-12=-5$

The equation is $\boldsymbol{y}=\mathbf{4 x} \mathbf{- 5}$

## Your Turn:

1. Complete the table:

| Equation | Gradient | $y$-intercept |
| :---: | :---: | :---: |
| $y=3 x+7$ |  |  |
| $y=2-x$ |  |  |
| $2 y=x+5$ | -2 | 0 |
| $3 y+2 x=-1$ | $\frac{3}{4}$ | -1 |
|  |  |  |

2. Find the equation of the line passing through the points with coordinates $(2,3)$ and $(4,1)$.
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3. A line whose gradient is $\frac{1}{3}$ passes through the point $(-6,9)$. Work out the equation of this line, giving your answer in the form $a y+b x=c$, where $a, b$ and $c$ are integers.
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4. The diagram shows two straight lines. Are the lines perpendicular? Justify your answer.

5. Does the line with equation $2 x+5 y=-1$ pass through the point with coordinates $(2,-1)$ ?
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$\qquad$
$\qquad$
$\qquad$

## For answers, go to page 108.

## Quadratic Graphs

All quadratic graphs are parabolas: a symmetrical curve. The orientation of this parabola depends on whether the coefficient of $x^{2}$ is positive or negative.


If the coefficient of $x^{2}$ is negative, the graphs are $n$-shaped:


We can use our skills for working with quadratic equations to find other key features of the graph:

1. The $y$-intercept is found by setting $x=0$ and solving the given equation for $y$.
2. The $x$-intercept is found by setting $y=0$ and solving the given equation for $x$.
3. The turning point (which will be the minimum or maximum of a quadratic function) can be found by completing the square. This can also be found by finding the average of the $x$-intercept values but completing the square can be more efficient. A graph with an equation of the form $y=a(x+b)^{2}+c$ has a turning point at $(-b, c)$.

For example,
Sketch the graph of $y=x^{2}-7 x+10$, clearly indicating any points of intersection with the axes and the location of the turning point of the curve.

The coefficient of $x^{2}$ is 1 . This is positive, so our graph will be $u$-shaped.
The $y$-intercept is found by substituting $x=0$ into the equation.
$y=0^{2}-7 \times 0+10$
$y=10$

The $x$-intercept is found by substituting $y=0$ into the equation then solving the resulting equation.
$0=x^{2}-7 x+10$
$0=(x-2)(x-5)$
$x=2, x=5$

The turning point is found by completing the square.

$$
\begin{aligned}
x^{2}-7 x+10 & =\left(x-\frac{7}{2}\right)^{2}-\left(\frac{7}{2}\right)^{2}+10 \\
& =\left(x-\frac{7}{2}\right)^{2}-\frac{9}{4}
\end{aligned}
$$

The turning point has coordinates $\left(\frac{7}{2},-\frac{9}{4}\right)$.

When sketching a graph, it does not need to be to scale but should be the right shape and roughly in proportion as shown:


## Your Turn:

1. Consider the curve with equation $y=x^{2}+4 x-5$.
a. Find the coordinates of the point where this curve intersects the $y$-axis.
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$\qquad$
b. Find the coordinates of the points where this curve intersects the $x$-axis.
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$\qquad$
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c. Hence, sketch the graph of $y=x^{2}+4 x-5$, clearly indicating any points of intersection with the axes.
$\square$
2. Consider the curve with equation $y=x^{2}+8 x-1$.
a. Find the coordinates of the turning point of this curve.
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$\qquad$
$\qquad$
b. State whether the turning point is a maximum or minimum. Justify your answer.
3. Sketch the graph of $y=x^{2}+4 x-21$, clearly indicating any points of intersection with the axes and the location of the turning point of the curve.
4. Sketch the graph of $y=-x^{2}+7 x$, clearly indicating any points of intersection with the axes.
5. Sketch the graph of $y=2 x^{2}+17 x+8$, clearly indicating any points of intersection with the axes.

## Quadratic Equations and Inequalities

To solve quadratic equations and inequalities, begin by checking if the quadratic part can be factorised. If it can, then this is the most efficient method to use.

## Example 1

Solve $2 x^{2}+9 x+7=0$
Start by factorising the quadratic part to get:
$(2 x+7)(x+1)=0$

For this equation to be correct, either of the expressions in the brackets must be equal to zero.
$2 x+7=0$ or $x+1=0$

Solve each linear equation to get $\boldsymbol{x}=-\frac{7}{2}$ or $\boldsymbol{x}=\mathbf{- 1}$.

We can always check the solutions by substituting them back into the original equation and making sure the result is zero.

We need to be a little bit careful when solving quadratic inequalities. We begin by using the same method: Set the quadratic part equal to zero and solve. This tells us where the graph crosses the $x$-axis. By sketching the quadratic curve, we can then decide which set of $x$-values makes the inequality true.

## Example 2

Solve $x^{2}+x-20>0$

Set equal to zero and solve:
$x^{2}+x-20=0$
$(x+5)(x-4)=0$
$x=-5$ or $x=4$

The graph of the equation $y=x^{2}+x-20$ looks like this:


The function is greater than zero for the parts of the curve that lie above the $x$-axis. That is when $x<-5$ and $x>4$.

We can also complete the square to help us find exact solutions to quadratic equations that cannot be factorised.

## Example 3

Solve $x^{2}-6 x+1=0$

Begin by completing the square for the quadratic part.
$(x-3)^{2}-10=0$

Next, solve by performing a series of inverse operations.
$(x-3)^{2}=10$
When we take the square root, we must find both the positive and negative square root of the numerical part.
$x-3= \pm \sqrt{10}$
$x=3 \pm \sqrt{10}$
The two solutions are $\boldsymbol{x}=\mathbf{3}+\sqrt{\mathbf{1 0}}$ and $\boldsymbol{x}=\mathbf{3 - \sqrt { 1 0 }}$

## Your Turn:

1. Solve by factorising:
a. $x^{2}-11 x+28=0$
c. $y^{2}+4 y-35=2 y$
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$\qquad$
$\qquad$
$\qquad$
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b. $3 x^{2}-16 x-12=0$
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$\qquad$
$\qquad$
d. $6 n^{2}-6 n=-n^{2}$
2. Solve the inequalities by sketching the graph: a. $x^{2}+12 x+32 \geq 0$
b. $x^{2}-2 x-8<-x-2$
c. $4 x^{2}+20 x-11 \leq 0$
d. $8 x^{2}+9 x>2 x$

Quadratic Equations and Inequalities
3. Solve by completing the square, writing surds in their simplest form:
a. $x^{2}+8 x+3=0$
c. $2 x^{2}+12 x+1=0$
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b. $x^{2}+3 x-7=0$
d. $x^{2}+16 x+3=-9$
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For answers, go to page 112.

## The Quadratic Formula

For quadratic equations that cannot be factorised, the quadratic formula can help. For an equation of the form $a x^{2}+b x+c=0$, the solutions are given by:
$x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$

Note that, since the equation includes a square root, there won't always be real solutions to a quadratic equation. The part inside the square root sign is called the discriminant and is referred to as the delta symbol $\Delta$. It follows that if this is less than zero then there are no real solutions to the equation. If it is equal to zero, there is one solution, and if it is greater than zero, there are two.

## Example 1

Solve the equation $3 x^{2}+5 x-4=0$, giving your answers correct to 3 significant figures.
$a=3, b=5$ and $c=-4$
$x=\frac{-5 \pm \sqrt{5^{2}-4 \times 3 \times-4}}{2 \times 3}$
$\boldsymbol{x}=\mathbf{- 2 . 2 6}$ or $\boldsymbol{x}=\mathbf{0 . 5 9 1}$

## Your Turn:

1. Solve $2 x^{2}+5 x+1=0$, giving your answers correct to 3 significant figures.
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$\qquad$
2. Solve $5 x^{2}+2 x=19$, giving your answers correct to 3 significant figures.
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$\qquad$
$\qquad$
3. Explain why the graph of the equation $y=x^{2}+4 x+9$ does not intersect the $x$-axis.

Use the discriminant $\Delta=b^{2}-4 a c$
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$\qquad$
$\qquad$
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$\qquad$

For answers, go to page 115.

## Simultaneous Equations

When solving simultaneous equations, we are finding the values of the given variables (usually $x$ and $y$ ) that make both equations true. There are two techniques: elimination (this works better for linear equations) and substitution (this works better for quadratic equations but can be used in either case).

## Example 1

Solve the simultaneous equations $2 x-4 y=-14$ and $3 x+7 y=18$.

To use the elimination method, we need to multiply one or both equations by some constant to create a common coefficient.

Let's call $2 x-4 y=-14$ equation 1 and $3 x+7 y=18$ equation 2 .

Multiply equation 1 by 3 and equation 2 by 2 .
$6 x-12 y=-42$
$6 x+14 y=36$

To eliminate the $x$ parts, subtract one equation from the other (if the signs of the coefficients are different then we add the equations) and then solve the resulting equation. It doesn't matter which way around we do this, as long as we are careful with the signs.

$$
\begin{aligned}
-26 y & =-78 \\
y & =3
\end{aligned}
$$

Now, substitute this value into either of the original equations and solve for $x$. Take equation 1:

$$
\begin{aligned}
2 x-4 \times 3 & =-14 \\
2 x-12 & =-14 \\
2 x & =-2 \\
x & =-1
\end{aligned}
$$

The solution is $x=-1$ and $y=3$.

## Example 2

Solve the simultaneous equations $x^{2}+y^{2}=5$ and $y-x=3$.

To use the substitution method, rearrange the linear equation to make $y$ the subject. $y=3+x$

Now, substitute this into the quadratic equation then solve.
$x^{2}+(3+x)^{2}=5$
$x^{2}+9+6 x+x^{2}=5$
$2 x^{2}+6 x+9=5$
$2 x^{2}+6 x+4=0$

We can factorise from here but it is easier to divide through by 2 , then factorise.
$x^{2}+3 x+2=0$
$(x+2)(x+1)=0$
$x=-2, x=-1$

Don't forget to substitute both of these back into the original linear equation to find the corresponding values for $y$.

When $x=-2$ :
$y-(-2)=3$
$y+2=3$
$y+1=3$
$y=1$
$y=2$

The solutions are $\boldsymbol{x}=-2, y=1$ and $x=-1, y=2$.

## Your Turn:

1. Solve each pair of simultaneous equations:
a. $x+y=14$
$2 x-y=16$
c. $2 x+5 y=26$
$y=x+1$
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b. $3 x+2 y=-4$
$2 x+y=-3$
$\qquad$
d. $x^{2}+y^{2}=10$
$y=x+2$
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e. $2 x^{2}=y^{2}-8$
$y-x=2$
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For answers, go to page 116.

