Number

At A level, there are no non-calculator papers and you are encouraged to use a calculator, where possible, to solve problems (including using calculators that can solve linear and polynomial equations).

You might think this makes a “Number” section pointless, but the tools included here become increasingly important. The line between what is a “Number” problem and what is an “Algebra” problem becomes very blurred, so you will need to be comfortable combining indices, surds and algebra.

At the same time, as the problems you solve become more complex, maintaining accuracy becomes more important. You’ll need to be comfortable working with intermediate answers as surds (or fractions, or in terms of π) to put off rounding until you have a final answer.

Negative and Zero Powers

An index is an instruction telling you how many times to multiply a number by itself, e.g. $5^3 = 5 \times 5 \times 5$. However, that index doesn't have to be a positive whole number; it can also be negative, zero or even a fraction. Look at the pattern of these indices to see how this works:

$$\begin{array}{ccccc}
 & 2^{3} & = & 2 \times 2 \times 2 & \\
\times 2 & & & & \div 2 \\
 & 2^{2} & = & 2 \times 2 & \\
\times 2 & & & & \div 2 \\
 & 2^{1} & = & 2 & \\
\times 2 & & & & \div 2 \\
 & 2^{0} & = & 1 & \\
\times 2 & & & & \div 2 \\
 & 2^{-1} & = & \frac{1}{2} & \\
\times 2 & & & & \div 2 \\
 & 2^{-2} & = & \frac{1}{2 \times 2} & \\
\times 2 & & & & \div 2 \\
 & 2^{-3} & = & \frac{1}{2 \times 2 \times 2} &
\end{array}$$

The first thing to notice is that $2^0 = 1$. Anything with an index of 0 is 1, whether the base is large, small or algebraic.

Example 1

$0.5^0 = \mathbf{1}$

$19\ 251\ 215^0 = \mathbf{1}$

$(3x^2 + 8x + 10)^0 = \mathbf{1}$

Secondly, when we are dealing with negative indices, we are effectively dividing instead of multiplying.

$2^3 = \mathbf{8}$

and $2^{-3} = \frac{1}{2^3} = \mathbf{\frac{1}{8}}$

This means that a negative power tells us to find the reciprocal of the number. Remember, the reciprocal of a number is what we get when we divide 1 by that number:

The reciprocal of x is $\frac{1}{x}$.

If the number is already a fraction, this is the same as inverting the numerator and denominator: The reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$.

Example 2

Evaluate 4^{-3}

The negative tells us to find the reciprocal. The reciprocal of 4 is $\frac{1}{4}$.

$4^{-3} = (\frac{1}{4})^3 = \frac{1}{4^3} = \mathbf{\frac{1}{64}}$

Notice that we are cubing both parts of the fraction, but since 1^3 is 1, we just write 1.

Likewise, if the base is a fraction, we find the reciprocal and then apply the index to both the numerator and denominator.

Example 3

a. Evaluate $(\frac{3}{5})^{-2}$

The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

$(\frac{3}{5})^{-2} = \frac{5^2}{3^2} = \mathbf{\frac{25}{9}}$

b. Evaluate $(\frac{1}{2})^{-2}$

$(\frac{1}{2})^{-2} = \frac{2^2}{1^2} = \frac{4}{1} = \mathbf{4}$

c. Evaluate $(\frac{2}{5})^{-3}$

$(\frac{2}{5})^{-3} = \frac{5^3}{2^3} = \mathbf{\frac{125}{8}}$

Your Turn:

1. a. What is $3^5 \div 3^2$ in index form?

   ____

   b. What is $3^2 \div 3^2$ in index form?

   ____

   c. Evaluate $3^2 \div 3^2$

   ____

   d. Evaluate 3^0

   ____

   e. Evaluate 27.54^0

   ____

   f. Evaluate $2.7523^0 \times 268^1 \times 892^0$

   ____

2. Evaluate the following:

   a. 5^{-2} __________

   b. 8^{-2} __________

   c. 3^{-3} __________

   d. 2^{-5} __________

3. Write each in index form:

a. $\frac{1}{16}$ ______ b. $\frac{1}{49}$ ______ c. $\frac{1}{125}$ ______ d. $\frac{1}{1000}$ ______

4. Evaluate, giving your answers as fractions in their simplest form:

a. $(\frac{3}{5})^{-1}$ ______ b. $(\frac{7}{8})^{-2}$ ______ c. $(\frac{1}{4})^{-3}$ ______ d. $(\frac{2}{3})^{-3}$ ______

5. Evaluate, giving your answers as fractions in their simplest form:

a. $(3x)^{-2}$ ______ b. $(2x^3)^{-2}$ ______ c. $(5x^4)^{-3}$ ______ d. $(2x^2y^3)^{-4}$ ______

For answers, go to page 92.

Fractional Powers

Fractional powers (fractional indices) are a way of representing a combination of roots or powers. The denominator represents the value of the root.

For example, a fractional power of $\frac{1}{2}$ represents a square root and $\frac{1}{3}$ represents a cube root.

Example 1

$25^{\frac{1}{2}} = \sqrt{25} = \mathbf{5}$

$8^{\frac{1}{3}} = \sqrt[3]{8} = \mathbf{2}$

$$(4x^2)^{\frac{1}{2}} = \sqrt{4x^2}$$
$$= \sqrt{4}\sqrt{x^2}$$
$$= \mathbf{2x}$$

The numerator of a fractional power represents an integer power, which allows a fractional power to represent both a power and a root.

For example, a fractional power of $\frac{3}{2}$ would tell you to raise the base to a power of 3, and to square root the answer. These two operations can be done in either order; you can use your judgement to decide which is easier.

Example 2

$$4^{\frac{3}{2}}$$
$$= \sqrt{4^3}$$
$$= \sqrt{64}$$
$$= \mathbf{8}$$

or $4^{\frac{3}{2}}$

$$= (\sqrt{4})^3$$
$$= 2^3$$
$$= \mathbf{8}$$

As you can see, the answer is the same whether you evaluate the power or the root first.

Example 3

$$9^{\frac{3}{2}}$$
$$= \sqrt{9^3}$$
$$= \sqrt{729}$$
$$= \mathbf{27}$$

When this is combined with negative powers, you are likely to have three operations to carry out. Again, the order will not change the result:

E.g. $(\frac{9}{25})^{-\frac{3}{2}} = (\frac{25}{9})^{\frac{3}{2}}$

$$= (\frac{\sqrt{25}}{\sqrt{9}})^3$$
$$= \frac{5^3}{3^3}$$
$$= \mathbf{\frac{125}{9}}$$

Your Turn:

1. Evaluate the following:

   a. $36^{\frac{1}{2}}$ b. $1000^{\frac{1}{3}}$ c. $64^{\frac{1}{3}}$ d. $81^{-\frac{1}{2}}$

2. Evaluate the following:

   a. $27^{\frac{2}{3}}$ b. $8^{\frac{4}{3}}$ c. $49^{\frac{3}{2}}$ d. $64^{\frac{2}{3}}$

3. Express in the form $a^{\frac{m}{n}}$, where m and n are integers.

   a. $\sqrt{a^3}$

   b. $\sqrt[3]{a^5}$

   c. $\frac{1}{\sqrt{a^7}}$

   d. $\sqrt{a} \times \frac{1}{\sqrt{a^5}}$

4. Write the following expressions in order, from smallest to largest:

   $25^{\frac{1}{2}}$, $8^{\frac{2}{3}}$, $27^{\frac{1}{3}}$, $(\frac{1}{9})^{-\frac{3}{2}}$, $(\frac{1}{12})^{-1}$, $(27^{\frac{5}{3}})^0$,

5. Write $64^{\frac{2}{3}} \times 2^3$ in the form 2^a, where a is a positive integer.

For answers, go to page 93.

Index Laws

The laws of indices are shortcuts for simplifying expressions involving indices without evaluating them.

Index Law for Multiplication: $a^x \times a^y = a^{x+y}$

E.g. Simplify $3^5 \times 3^2$. Give your answer in index notation.

Firstly, remember what an index is – an instruction to multiply a number by itself.

$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$3^2 = 3 \times 3$

$3^5 \times 3^2 = \underline{3 \times 3 \times 3 \times 3 \times 3} \times \underline{3 \times 3} = \mathbf{3^7}$

The index law for multiplication lets you skip the intermediate steps and go straight to $3^5 \times 3^2 = 3^7$

It's important to remember that all three of these laws work with both numerical and algebraic bases. Read the question carefully. If it says evaluate, you are expected to give a numerical answer without an index. If it says index notation, you are expected to give it in the form a^b.

E.g.: $x^3 \times x^5 = x^8$

$5^a \times 5^{2b} = \mathbf{5^{a+2b}}$

You can only use these laws if both bases are the same:

E.g.: $3^5 \times 5^2 \neq 15^7$

$a^5 \times b^3 \times a^2 \times b^3 = \mathbf{a^7 \times b^6}$

Index Law for Division: $a^x \div a^y = a^{x-y}$

E.g. $a^7 \div a^4$

$$= \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a}$$

$$= \frac{a \times a \times a \times \cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}}$$ (cancel out where possible)

$= \mathbf{a^3}$

Or, using the index law as a shortcut: $a^7 \div a^4 = a^{7-4} = \mathbf{a^3}$

Again, remember that this will work with both numerical and algebraic terms. Be very careful with negative numbers. It is easy to make mistakes when dealing with negative indices.

E.g.: $5^4 \div 5^{-3} = 5^{4--3} = 5^{4+3} = \mathbf{5^7}$

$x^{-5} \div x^7 = x^{-5-7} = \mathbf{x^{-12}}$

Index Law for Powers: $(a^x)^y = a^{x \times y}$

E.g. $(5^3)^4$

$= \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5}$

$= \mathbf{5^{12}}$

Or, using the index law for powers: $(5^3)^4 = 5^{3 \times 4} = 5^{12}$

Again, be careful of negative numbers. Sometimes, you will have to apply more than one of these laws, the order will not change the result. Which order you apply them is up to your judgement.

E.g.: $\left(\frac{(a^4 \times a^3)}{(a^2)}\right)^2$

$= \left(\frac{(a^7)}{(a^2)}\right)^2$ Use the index law for multiplication on the numerator.

$= (a^5)^2$ Then use the index law for division to remove the fraction.

$= \boldsymbol{a^{10}}$ Then use the index law for powers to get a final answer.

Your Turn:

1. Simplify each expression. Give your answers in index form.

   a. $5^4 \times 5^8$ ______ b. $m^4 \div m^2$ ______ c. $(a^3)^2$ ______ d. $3^5 \times 3$ ______

2. Simplify each expression. Give your answers in index form.

   a. $3^8 \times 3^{-2}$ ______ b. $\frac{h^{-3}}{h^5}$ ______ c. $p^{-2} \div p^{-9}$ ______ d. $(5^{-3})^{-2}$ ______

3. Simplify each expression. Give your answers in index form.

   a. $3a^2 \times 3a^5$ ______ b. $(3x^4)^3$ ______ c. $\frac{12x^3}{4x^5}$ ______ d. $a^2b^5 \times a^4b^{-8}$ ______

4. Simplify the expression. Give your answer in index form.

   $\left(\frac{3a^5 \times 6a^{-7}}{2a^5}\right)^2$ ______

For answers, go to page 95.

Simplifying Surds

When you find the square root of an integer, your answer will be one of two types of number. If you find the root of a square number, your result will be an integer: $\sqrt{4} = 2$; if you find the square root of any other integer, your answer will be irrational: $\sqrt{2} = 1.414213562373095...$

A rational number is any number that can be represented as a fraction of two integers, e.g. $\frac{1}{3}$ (even though it recurs) or $\frac{3}{1}$ (the integer 3), while an irrational number is any number that is not rational.

An irrational number will never end and never repeat so if a square root results in an irrational number, you either have to round your answer to write it, or leave it as a root.

If you round it, you have lost some of the accuracy of the answer. A surd is therefore an irrational number that has been left in the form of a root to represent its exact value.

For example, we want to find the value of $5x^2 + 3$, where $x = \sqrt{6}$.

We have two options: 1. Use $x = \sqrt{6}$;

2. Use $x = 2.4$ ($\sqrt{6}$ rounded to 1d.p.)

1. $5(\sqrt{6})^2 + 3 = 5 \times 6 + 3 = \mathbf{33}$
2. $5 \times 2.4^2 + 3 = 5 \times 5.76 + 3 = \mathbf{31.8}$

Even with an example with small numbers, you can see how different the rounded answer is. In any situation dealing with roots, it is therefore important that you can operate with numbers in surd form.

To do this, you need to be able to simplify surds, starting with multiplying and dividing. Here are some examples:

E.g. a. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$

$= \sqrt{\mathbf{6}}$

b. $2\sqrt{5} \times 4\sqrt{2} = 2 \times 4 \times \sqrt{5 \times 2}$

$= \mathbf{8\sqrt{10}}$

c. $\sqrt{10} \div \sqrt{5} = \sqrt{10 \div 5}$

$= \sqrt{\mathbf{2}}$

d. $\sqrt{10} \div \sqrt{3} = \sqrt{\mathbf{\frac{10}{3}}}$

When multiplying surds, you simply multiply the base of each root.

If there are coefficients outside the root, multiply them separately.

You can divide the bases if the result is an integer. If the result is not an integer, you will normally write the surds as a fraction.

Your Turn:

1. a. $\sqrt{5} \times \sqrt{7}$

   b. $3\sqrt{2} \times 4\sqrt{5}$

   c. $\sqrt{15} \div \sqrt{3}$

   d. $18\sqrt{20} \div 6\sqrt{5}$

   e. $5\sqrt{2} \times 3\sqrt{8}$

   f. $2\sqrt{3} \times 5$

   g. $(5\sqrt{3})^2$

   h. $(2\sqrt{5})^3$

2. A right-angled triangle has a height of $6\sqrt{5}$cm and a base of $7\sqrt{3}$cm. Find its area.

For answers, go to page 96.

Addition and subtraction of surds can be more complex. In the same way you cannot add or subtract fractions with different denominators, you cannot simply add or subtract surds with different bases:

E.g. a. $3\sqrt{5} + 8\sqrt{5} = 11\sqrt{5}$

b. $2\sqrt{5} + 8\sqrt{3} \neq 10\sqrt{8}$

In some cases, such as example b above, the addition is not possible and you would leave your answer as $2\sqrt{5} + 8\sqrt{3}$. In other cases, you can simplify one surd so it has the same base as the other:

E.g. $\sqrt{2} + \sqrt{8}$

Initially, this might look impossible. However, 8 has a square factor (4 × 2). This means:

$\sqrt{8}$

$= \sqrt{4 \times 2}$

$= \sqrt{4}\sqrt{2}$

$= \mathbf{2\sqrt{2}}$

The key to simplification is to find a square factor. Since we know $\sqrt{4} = 2$, we can write $\sqrt{8}$ as $2\sqrt{2}$, giving:

$\sqrt{2} + \sqrt{8}$

$= \sqrt{2} + 2\sqrt{2}$

$= \mathbf{3\sqrt{2}}$

Here's another example:

$3\sqrt{3} + 2\sqrt{12}$

$= 3\sqrt{3} + 2\sqrt{3 \times 4}$ Look for a square factor.

$= 3\sqrt{3} + 2(\sqrt{4}\sqrt{3})$

$= 3\sqrt{3} + 2(2\sqrt{3})$

$= 3\sqrt{3} + 4\sqrt{3}$

$= \mathbf{7\sqrt{3}}$

Your Turn:

1. Simplify these surds (remember: the key is to find a square factor).
   a. $\sqrt{20}$ ______ b. $\sqrt{48}$ ______ c. $\sqrt{75}$ ______ d. $5\sqrt{8}$ ______

2. Answer the following, giving your answers in the form $a\sqrt{b}$
   a. $\sqrt{2} + \sqrt{18}$ ______ b. $\sqrt{50} - \sqrt{200}$ ______

c. $4\sqrt{80} + 3\sqrt{45}$

d. $2\sqrt{50} + 5\sqrt{32}$

3. A rectangle has a width of $6\sqrt{75}$m and a height of $2\sqrt{12}$m. What is its perimeter?

Challenge

A right-angled triangle has a base of $2\sqrt{18}$cm and a height of $2\sqrt{32}$cm. Find the perimeter of the triangle.

For answers, go to page 96.

Rationalising the Denominator

A surd is a way of expressing an irrational root without losing accuracy. When a surd is on the denominator of a fraction, this fraction can be simplified by replacing that surd with an integer. This is called "rationalising" the denominator.

Consider this fraction:

$$\frac{1}{\sqrt{2}}$$

As 2 is not a square number, $\sqrt{2}$ is irrational and we want to remove it from the denominator. We can do this by multiplying our fraction by $\frac{\sqrt{2}}{\sqrt{2}}$.

As $\frac{\sqrt{2}}{\sqrt{2}}$ cancels to 1, this multiplication does not change the value of the fraction.

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}}$ Multiply the numerators, multiply the denominators

$= \mathbf{\frac{\sqrt{2}}{2}}$ $(\sqrt{2} \times \sqrt{2} = 2)$

There is now a surd in the numerator but the denominator is a rational number, 2. Here are some more examples:

E.g. a. Rationalise the denominator of $\frac{2}{\sqrt{3}}$

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \mathbf{\frac{2\sqrt{3}}{3}}$$

b. Rationalise the denominator of $\frac{7}{\sqrt{6}}$

$$\frac{7}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \mathbf{\frac{7\sqrt{6}}{6}}$$

Your Turn:

1. Rationalise the denominator of each fraction.

a. $\frac{5}{\sqrt{2}}$ ____________

c. $\frac{\sqrt{2}}{\sqrt{3}}$ ____________

e. $\frac{2-\sqrt{3}}{\sqrt{3}}$ ____________

b. $\frac{4}{\sqrt{3}}$ ____________

d. $\frac{3}{2\sqrt{5}}$ ____________

2. Rationalise the denominator of $\frac{\sqrt{5}}{\sqrt{80}}$. Give your answer as a fraction in its simplest form.

3. What is $\frac{2\sqrt{2}}{\sqrt{6}} + \frac{1}{\sqrt{3}}$? Give your answer in its simplest terms.

For answers, go to page 98.

A similar method can be used to rationalise fractions with more complicated denominators, such as:

$$\frac{2}{\sqrt{3} + 4}$$

In this case, multiplying numerator and denominator by $\sqrt{3}$ will give us:

$$\frac{2\sqrt{3}}{3 + 4\sqrt{3}}$$

This has not rationalised the denominator - we need to multiply by something different.

Consider what happens when you expand a pair of brackets in the form $(a + b)(a - b)$:

$$(a + b)(a - b)$$
$$= a^2 + ab - ab - b^2$$
$$= \boldsymbol{a^2 - b^2}$$

In this case, all we are left with is the difference of two squares. If a or b were surds, they would now be rational.

We can apply this technique to rationalising denominators.

$\frac{2}{\sqrt{3} + 4} \times \frac{\sqrt{3} - 4}{\sqrt{3} - 4}$	Multiply numerator and denominator by $\sqrt{3} - 4$. Notice we have changed the sign on the denominator (this is called the conjugate).
$= \frac{2\sqrt{3} - 8}{(\sqrt{3} + 4)(\sqrt{3} - 4)}$	Start by expanding the brackets in the numerator
$= \frac{2\sqrt{3} - 8}{\sqrt{3}\sqrt{3} + 4\sqrt{3} - 4\sqrt{3} - 16}$	Then, expand the brackets in the denominator
$= \boldsymbol{\frac{2\sqrt{3} - 8}{-13}}$ or $\boldsymbol{\frac{8 - 2\sqrt{3}}{13}}$	Finally, simplify the denominator. Remember: $\sqrt{3} \times \sqrt{3} = 3$

The surd is now rationalised.

Your Turn:

4. a. $\frac{5}{\sqrt{2} + 7}$ b. $\frac{1}{\sqrt{5} - 3}$ c. $\frac{1 + \sqrt{2}}{\sqrt{3} + 2}$

Challenge

Amy is laying tiles in her rectangular bathroom. By the time she has finished, she has used $8m^2$ worth of tiles. She knows the length of one side of the room is $(\sqrt{5} + 2)$m but, unfortunately, she has lost her tape measure. Amy still needs to work out the perimeter of the room. Calculate the perimeter of the room, giving your answer in its simplest form.

For answers, go to page 98.